TWISTED TRACES OF SINGULAR MODULI OF WEAKLY HOLOMORPHIC MODULAR FUNCTIONS

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ABSTRACT. Zagier proved that the generating series for the traces of singular moduli is a *weakly holomorphic* modular form of weight 3/2 on $\Gamma_0(4)$. Bruinier and Funke extended the results of Zagier to modular curves of arbitrary genus. Zagier also showed that the twisted traces of singular moduli are generated by a weakly holomorphic modular form of weight 3/2. In this paper, we study the extension of Zagier's result for the twisted traces of singular moduli to congruence subgroups $\Gamma_0(N)$. As an application, we study congruences for the twisted traces of singular moduli of weakly holomorphic modular functions on $\Gamma_0(N)$.

1. Introduction

Let $j(z)$ be the usual j-invariant function defined for z in the complex upper half plane $\mathbb{H}$ by $j(z) = q^{-1} + 744 + 196884q + \cdots$, where $q = e(z) = e^{2\pi i z}$. The function $J(z) = j(z) - 744$ is the normalized Hauptmodul for the group $\Gamma(1) = PSL_2(\mathbb{Z})$. For a positive integer D congruent to 0 or 3 modulo 4, denote by $\mathcal{Q}_D$ the set of positive definite integral binary quadratic forms

$$Q(x, y) = [a, b, c] = ax^2 + bxy + cy^2$$

with discriminant $-D = b^2 - 4ac$. The group $\Gamma(1)$ acts on $\mathcal{Q}_D$ by $Q \circ \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) = Q(\alpha x + \beta y, \gamma x + \delta y)$. For each $Q \in \mathcal{Q}_D$ let

$$z_Q = \frac{-b + i\sqrt{D}}{2a},$$

the corresponding CM point in $\mathbb{H}$ and $\Gamma(1)_Q$ denote the stabilizer of Q in $\Gamma(1)$. The values of j or other modular functions at CM points z_Q are known as *singular moduli*, and they play important roles in number theory. For example, if $-D$ is a negative fundamental discriminant, then $j(z_Q)$ generates the Hilbert class field of $\mathbb{Q}(\sqrt{-D})$ (see [8]). We define the trace of singular moduli of index D by

$$\mathbf{t}_J(D) = \sum_{Q \in \mathcal{Q}_D/\Gamma(1)} \frac{1}{|\Gamma(1)_Q|} J(z_Q). \tag{1.1}$$

2000 *Mathematics Subject Classification.* 11F27, 11F37.

Key words and phrases. modular traces, theta liftings.

In [23, Theorem 1], Zagier proved that the generating series for the traces of singular moduli

$$g(z) := q^{-1} - 2 - \sum_{\substack{D>0 \\ D\equiv 0,3 (\mathrm{mod}\ 4)}} \mathbf{t}_J(D) q^D = q^{-1} - 2 + 248q^3 - 492q^4 + \cdots \tag{1.2}$$

is a weakly holomorphic modular form of weight 3/2 on $\Gamma_0(4)$, that is, holomorphic on $\mathbb{H}$ and meromorphic at each cusp. The results of Zagier were extenend in [11] and [12] to congruence subgroups $\Gamma_0(N)$ of genus zero with prime levels N.

Bruinier and Funke [3] generalized the results of Zagier to the traces of singular moduli of modular functions on congruence subgroups of arbitrary genus. They proved that the generating series for the traces of CM values of a weakly holomorphic modular function on a modular curve of arbitrary genus is given by the holomorphic part of a harmonic weak Maass form of weight 3/2. In [22], Zagier defined the twisted traces of singular moduli for $J(z)$ and proved that the generating series for the twisted traces of singular moduli is also a weakly holomorphic modular form of weight 3/2. In this paper, using the method of Bruinier and Funke [3], we study modularity of the twisted traces of CM values of weakly holomorphic modular functions and their congruences.

Following the definition of Zagier [22], we define the twisted traces of a weakly holomorphic modular function on $\Gamma_0(N)$. For a positive integer N, let $\mathcal{Q}_{D,N}$ be the set of quadratic forms $Q \in \mathcal{Q}_D$ such that $a \equiv 0 \pmod N$. We note that $-D$ is congruent to a square modulo $4N$ and the group $\Gamma_0(N)$ acts on $Q_{D,N}$ with finitely many orbits, where the action of $\Gamma_0(N)$ is defined as above. Let $\Gamma_0(N)_Q$ be the stabilizers of Q in $\Gamma_0(N)$. Let $\Delta \in \mathbb{Z}$ be a fundamental discriminant and $r \in \mathbb{Z}$ such that $\Delta \equiv r^2 \pmod{4N}$. Following the definition in [10], we define a generalized genus character for $[Na, b, c] \in \mathcal{Q}_{D,N}$ as follows:

$$\chi_\Delta(X) = \chi_\Delta([Na,b,c]) = \begin{cases} \left(\frac{\Delta}{n}\right), & \text{if } \Delta | b^2 - 4Nac \text{ and } (b^2-4Nac)/\Delta \text{ is a square} \\ & \text{modulo } 4N \text{ and } \gcd(a,b,c,\Delta) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Here n is any integer prime to Δ represented by one of the quadratic forms $[N_1a, b, N_2c]$ with $N_1N_2 = n$ and $N_1, N_2 > 0$ (see 1.2 in [10]). If f is a weakly holomorphic modular function on $\Gamma_0(N)$, then the twisted trace of f of positive index D is defined by

$$\mathbf{t}_f(\chi_\Delta; D) = \sum_{Q \in \mathcal{Q}_{D,N}/\Gamma_0(N)} \chi_\Delta(Q) \cdot \frac{f(z_Q)}{|\overline{\Gamma_0(N)}_Q|}, \tag{1.3}$$

With these notations, we state our main theorem.

Theorem 1.1. *Suppose that N is a positive integer, and that $f(z)$ is a weakly holomorphic modular function on $\Gamma_0(N)$. If r is a sufficiently large integer, then for each positive odd*

integer t the function

$$\sum_{\substack{m>0 \\ \left(\frac{rm}{t}\right)=-1}} \mathbf{t}_f(\chi_\Delta; rm) q^m$$

is a weakly holomorphic modular form of weight $3/2$ on $\Gamma_1(4rt^2N)$. Here, $\left(\frac{m}{t}\right)$ denotes the Jacobi symbol.

As an application, we study congruence properties for the twisted traces of CM values of weakly holomorphic modular functions on $\Gamma_0(N)$. Ahlgren and Ono [1] studied divisibility of the traces of singular moduli in terms of the factorization of primes in imaginary quadratic fields. For example, they proved that for each positive integer ν, a positive proportion of primes r has the property that $\mathbf{t}_J(r^3n) \equiv 0 \pmod{p^\nu}$ for every positive integer n coprime to r such that p is inert or ramified in $\mathbb{Q}\left(\sqrt{-nr}\right)$. This result was extended in [20] and [7] to the traces of singular moduli of a weakly holomorphic modular function on $\Gamma_0(N)$ for any integer N. Here, $\Gamma_0^*(N)$ denotes the group extension of $\Gamma_0(N)$ by the group of Atkin-Lehner involutions W_p for all primes $p \mid N$. We obtain analogues of the results in [1], [20] and [7] for the twisted traces of CM values of weakly holomorphic modular functions.

Corollary 1.2. *Suppose that p is an odd prime, and N, $p \nmid N$, and t are a positive odd integer. Let K be an algebraic number field. Suppose that $f(\tau) = \sum a(n)q^n \in K((q))$ is a weakly holomorphic modular function on $\Gamma_0(N)$. Then there exists an integer Ω such that if m is sufficiently large, then for each positive integer ν, a positive proportion of primes $r \equiv -1 \pmod{4t^2Np^\nu}$ have the property that*

$$\Omega \mathbf{t}_f(\chi_\Delta; r^3p^mn) \equiv 0 \pmod{p^\nu}$$

for all n such that $\gcd(n, rpN) = 1$ and $\left(\frac{r^3p^mn}{t}\right) = -1$.

This paper is organized as follows. In section 2 we recall basic facts on real quadratic spaces and modular curves, and then define the twisted traces of CM values of a weakly holomorphic modular function on $\Gamma_0(N)$. In section 3, to prove the main theorems, we define a theta kernel and a theta lift for a weakly holomorphic modular functions on $\Gamma_0(N)$. In section 4 and 5 we give the proofs of the main theorems.

2. **Preliminaries**

For basic facts on rational quadratic spaces and modular curves, we refer to [3] and [7, Section 2] and follow notations in [3]. We consider a quadratic space (V, q) over $\mathbb{Q}$ of signature $(1, 2)$ given by

$$V(\mathbb{Q}) := \{X \in M_2(\mathbb{Q}) | \mathrm{tr}(X) = 0\}$$

with the associated quadratic form $q(X) := \det(X)$ and the bilinear form $(X, Y) := -\mathrm{tr}(XY)$. The group $SL_2(\mathbb{Q})$ acts on V by conjugation:

$$g \cdot X := gXg^{-1}$$

for $X \in V$ and $g \in SL_2(\mathbb{Q})$. This orthogonal transformation gives rise to an isomorphism $G := \mathrm{Spin}(V) \simeq SL_2$. We write $D := G(\mathbb{R})/K$ for the associated orthogonal symmetric space, where $K = SO(2)$. Then, we have $D \simeq \mathbb{H}$, the upper half plane. We may regard D as the space of positive lines in $V(\mathbb{R})$, that is, $D = \{\mathrm{span}(X) \subset V(\mathbb{R}) \,|\, (X, X) > 0\}$ and can give the following identification of D with $\mathbb{H}$. We pick as a base point of D the line spanned by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ so that $K = SO(2)$ is its stabilizer in $G(\mathbb{R})$. For $z = x + iy \in \mathbb{H}$, we choose $g_z \in G(\mathbb{R})$ such that $g_z i = z$, where the action is the usual linear fractional transformation on $\mathbb{H}$. We now have the isomorphism $\mathbb{H} \to D$ which assigns $z \in \mathbb{H}$ the positive line in D spanned by

$$X(z) := g_z \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \frac{1}{y} \begin{pmatrix} -\frac{1}{2}(z + \bar{z}) & z\bar{z} \\ -1 & \frac{1}{2}(z + \bar{z}) \end{pmatrix}.$$

Note that $q(X(z)) = 1$ and $g \cdot X(z) = X(gz)$ for $g \in G(\mathbb{R})$.

To define CM points in D, we need the following set-up. First, let $L \subset V(\mathbb{Q})$ be an even lattice of full rank and write $L^{\#}$ for the dual lattice of L. If Γ denotes a congruence subgroup of $\mathrm{Spin}(L)$ which preserves L and acts trivially on the discriminant group $L^{\#}/L$, then the attached locally symmetric space $M := \Gamma\backslash D$ is a modular curve, i.e., non-compact, as our space V is isotropic over $\mathbb{Q}$. The set $\mathrm{Iso}(V)$ of all isotropic lines in V corresponds to $P^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$, the set of cusps of $G(\mathbb{Q})$, via the bijective map $\psi : P^1(\mathbb{Q}) \to \mathrm{Iso}(V)$, which is defined by $\psi((\alpha : \beta)) = \mathrm{span} \begin{pmatrix} -\alpha\beta & \alpha^2 \\ -\beta^2 & \alpha\beta \end{pmatrix} \in \mathrm{Iso}(V)$. As ψ commutes with the $G(\mathbb{Q})$-actions, that is, $\psi(g(\alpha : \beta)) = g \cdot \psi((\alpha : \beta))$ for $g \in G(\mathbb{Q})$, the cusps of M, i.e., the Γ-classes of $P^1(\mathbb{Q})$, can be identified with the Γ-classes of $\mathrm{Iso}(V)$. In particular, the cusp $\infty \in P^1(\mathbb{Q})$ is mapped to the isotropic line ℓ_0 which is spanned by $X_0 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. We orient all lines $\ell \in \mathrm{Iso}(V)$ by regarding $\sigma_\ell \cdot X_0$ as a positively oriented basis vector of ℓ, where $\sigma_\ell \in SL_2(\mathbb{Z})$ such that $\sigma_\ell \cdot \ell_0 = \ell$. For each isotropic line $\ell \in \mathrm{Iso}(V)$, there exist positive rational numbers α_ℓ and β_ℓ such that $\sigma_\ell^{-1}\Gamma_\ell\sigma_\ell = \left\{ \pm \begin{pmatrix} 1 & k\alpha_\ell \\ 0 & 1 \end{pmatrix} \,\middle|\, k \in \mathbb{Z} \right\}$, where Γ_ℓ denotes the stabilizer of the line ℓ, and $\begin{pmatrix} 0 & \beta_\ell \\ 0 & 0 \end{pmatrix}$ is a primitive element of $\ell_0 \cap \sigma_\ell^{-1}L$, respectively. Finally, we write $\varepsilon_\ell = \alpha_\ell/\beta_\ell$. Note that α_ℓ is the width of the cusp ℓ of a congruence subgroup of $SL_2(\mathbb{Z})$ and the quantities α_ℓ, β_ℓ, and ε_ℓ only depend on the Γ-class of ℓ.

We denote the space of weakly holomorphic modular forms of weight k on Γ by $M_k^!(\Gamma)$. Let us define CM points as, for $X \in V(\mathbb{Q})$ of positive norm, i.e., $q(X) > 0$,

$$D_X = \operatorname{span}(X) \in D.$$

Note that the corresponding point in $\mathbb{H}$ satisfies a quadratic equation over $\mathbb{Q}$. Since the stabilizer G_X of X in $G(\mathbb{R})$ is isomorphic to $SO(2)$ which is compact, $\Gamma_X = G_X \cap \Gamma$ is finite. For $m \in \mathbb{Q}_{>0}$ and $h \in L^{\#}$, the group Γ acts on

$$L_{h,m} = \{X \in L + h \,|\, q(X) = m\}$$

with finitely many orbits. We define the *Heegner divisor* of discriminant m on M by

$$Z(h,m) = \sum_{X \in \Gamma \backslash L_{h,m}} \frac{1}{|\overline{\Gamma_X}|} D_X.$$

On the other hand, for a vector $X \in V(\mathbb{Q})$ of negative norm, we define a geodesic c_X in D by

$$c_X = \{z \in D \,|\, z \perp X\}.$$

We note from [9, Lemma 3.6] that the case $X^\perp \subset V(\mathbb{Q})$ is split over $\mathbb{Q}$ is equivalent to $q(X) \in -(\mathbb{Q}^\times)^2$. In that case, the stabilizer $\overline{\Gamma}_X$ is trivial, the quotient $c(X) := \Gamma_X \backslash c_X$ in M is an infinite geodesic, and X is orthogonal to the two isotropic lines $\ell_X = \operatorname{span}(Y)$ and $\tilde{\ell}_X = \operatorname{span}(\tilde{Y})$, with Y and $\tilde{Y}$ positively oriented. We say ℓ_X is the line associated to X if the triple $(X, Y, \tilde{Y})$ is a positively oriented basis for V, and we write $X \sim \ell_X$. Note $\tilde{\ell}_X = \ell_{-X}$.

If $m \in \mathbb{Q}_{>0}$ and $X \in L_{h,-m^2}$, then we can choose the orientation of V such that

$$\sigma_\ell^{-1} X = \begin{pmatrix} m & r \\ 0 & -m \end{pmatrix}$$

for some $r \in \mathbb{Q}$. The geodesic c_X is given in $D \simeq \mathbb{H}$ by

$$c_X = \sigma_{\ell_X} \{z \in \mathbb{H} \mid \Re(z) = -r/2m\}.$$

The number $-r/2m$ is called as the real part of the infinite geodesic $c(X)$ and denoted by $Re(c(X))$. We define

$$\langle f, c(X) \rangle = -\sum_{n<0} a_{\ell_X}(n) e^{2\pi i Re(c(X))n} - \sum_{n<0} a_{\ell_{-X}}(n) e^{2\pi i Re(c(-X))n}.$$

From now on, we define the twisted traces of CM values of a weakly holomorphic modular function on $\Gamma_0(N)$. Let N be a positive integer. Suppose that

$$L = \left\{ \begin{pmatrix} b & 2c \\ 2aN & -b \end{pmatrix}; a, b, c \in \mathbb{Z} \right\}$$

and $\Gamma = \Gamma_0(N)$. Let $\Delta \in \mathbb{Z}$ be a fundamental discriminant and $r \in \mathbb{Z}$ such that $\Delta \equiv r \pmod{4N}$. Following the definition in [10], we define a generalized genus character for $X = \left(\begin{smallmatrix} b & 2c \\ 2aN & -b \end{smallmatrix}\right) \in L$ as follows:

$$\chi_\Delta(X) = \chi_\Delta([Na, b, c]).$$

With these notations we define the twisted trace of a weakly holomorphic modular function f on Γ.

Definition 2.1. Suppose that $f(z)$ is a weakly holomorphic modular function on Γ. For a general genus character χ_Δ we define the twisted trace $\mathbf{t}_f(\chi_\Delta; m)$ as follows:

(1) If $m > 0$ and $m \in \mathbb{Q}_{>0}$, then

$$\mathbf{t}_f(\chi_\Delta; m) := \sum_{X \in \Gamma \backslash L_{0,m}} \frac{\chi_\Delta(X)}{|\overline{\Gamma}_X|} f(D_X). \tag{2.1}$$

(2) If $m = 0$, or $m < 0$ such that $m \notin -(\mathbb{Q}^\times)^2$, then

$$\mathbf{t}_f(\chi_\Delta; m) := 0. \tag{2.2}$$

(3) If $m < 0$ and $m \in -(\mathbb{Q}^\times)^2$, then

$$\mathbf{t}_f(\chi_\Delta; m) := \sum_{X \in \Gamma \backslash L_{0,m}} \chi_\Delta(X) \langle f, c(X) \rangle. \tag{2.3}$$

Remark 2.2. Note that the definition of twisted traces for positive index m in (2.1) is the same as the definition in (1.3).

3. Twisted theta kernels

Suppose that N is a positive integer and $\Gamma = \Gamma_0(N)$. In [13], Kudla constructed a Green function ξ^0 associated to a Poincare dual form $\varphi^0(X, z)$ for the Heegner point D_X. We recall the construction of ξ^0. Let

$$\mathrm{Ei}(w) = \int_{-\infty}^{w} \frac{e^t}{t} dt,$$

where the path of integration lies in the along the positive real axis (see [2]). For $X \in V(\mathbb{R})$, $X \neq 0$, we define

$$\xi^0(X, z) = -\mathrm{Ei}(-2\pi R(X, z)), \tag{3.1}$$

where

$$R(X, z) = \frac{1}{2}(X, X(z))^2 - (X, X).$$

It is known that $\mathrm{Ei}(\xi^0(X, z))$ is a smooth function on $D \setminus D_X$. For $q(X) > 0$, the function $\xi^0(X, z)$ has logarithmic growth at the point D_X, while it is smooth on D if $q(X) \leq 0$. Moreover, if $X \neq 0$, then away from the point D_X

$$\frac{1}{2\pi i} \cdot \overline{\partial}\partial \xi^0(X, z) = \varphi(X, z).$$

For $\tau = u + iv \in \mathbb{H}$, let

$$\varphi(X, \tau, z) = e^{2\pi i q(X)\tau} \varphi^0(\sqrt{v}X, z).$$

We define a theta kernel θ_Δ by

$$\theta_\Delta(\tau, z, \varphi) = \sum_{X \in L} \chi_\Delta(X)\varphi(X, \tau, z). \tag{3.2}$$

Since χ_Δ is invariant under the action of $\Gamma_0(N)$ and

$$\varphi^0(g.\sqrt{v}X, gz) = \varphi^0(\sqrt{v}X, z),$$

we have for $g \in \Gamma_0(N)$

$$\theta_\Delta(\tau, gz, \varphi) = \theta_\Delta(\tau, z, \varphi).$$

Proposition 3.1. *The theta kernel $\theta_\Delta(\tau, z, \varphi)$ is a non-holomorphic modular form of weight $3/2$ with values in $\Omega^{1,1}(M)$ on a congruence subgroup $\Gamma_1(4N)$. For each cusp ℓ we have*

$$\theta_\Delta(\tau, \sigma_\ell z, \varphi) = O(e^{-cy^2}) \text{ as } y \to \infty,$$

uniformly in x, for some constant $C > 0$.

Proof. Take

$$L_\Delta := \left\{ \Delta \begin{pmatrix} b & 2c \\ 2aN & -b \end{pmatrix}; a, b, c \in \mathbb{Z} \right\}.$$

For $h \in L_\Delta$ let

$$\theta_h(\tau, z, \varphi) = \sum_{X \in L_\Delta + h} \varphi(X, \tau, z).$$

Since $\chi_\Delta(X)$ depends on $X \in L$ modulo L_Δ, we have

$$\theta_\Delta(\tau, z, \varphi) = \sum_{h \in L/L_\Delta} \chi_\Delta(h) \sum_{X \in L_\Delta} \varphi(X, \tau, z) = \sum_{h \in L/L_\Delta} \chi_\Delta(h)\theta_h(\tau, z, \varphi). \tag{3.3}$$

It is known that $\theta_h(\tau, z, \varphi)$ is a non-holomorphic modular form of weight $3/2$ with values in $\Omega^{1,1}(M)$ for congruence subgroup $\Gamma(\Delta N)$, and that for each cusp ℓ we have

$$\theta_h(\tau, \sigma_\ell z, \varphi) = O(e^{-cy^2}) \text{ as } y \to \infty,$$

uniformly in x, for some constant $C > 0$ (see [9], Proposition 4.1).

Note that if $\Delta \nmid q(X)$, then $\chi_\Delta(X) = 0$. This implies that

$$\theta_\Delta(\tau, z, \varphi) = \sum_{n \in \mathbb{Z}} \sum_{q(X) = n\Delta} \chi_\Delta(X)\varphi^0(\sqrt{v}X, z)e^{2\pi i \Delta n \tau}.$$

Let $\Theta(\tau) := 1 + 2\sum_{n=1}^{\infty} q^{n^2}$. Note that the function $\Theta(\tau)$ is a modular form of weight $1/2$ on $\Gamma_0(4)$. For a positive integer m and a function $f(\tau)$ on $\mathbb{H}$, we define operators U_m and V_m by

$$f(\tau)|U_m = \frac{1}{m} \sum_{j=0}^{m-1} f\left(\frac{\tau + j}{m}\right) \text{ and } f(\tau)|V_m = f(m\tau).$$

Then

$$\theta_\Delta(\tau, z, \varphi)|U_\Delta = \sum_{n\in\mathbb{Z}} \sum_{q(X)=n\Delta} \chi_\Delta(X)\varphi^0(\sqrt{v/\Delta}X, z)e^{2\pi i n\tau}$$

(see Chapter I in [17] for details of operators U_m and V_m).Thus, we have

$$\Theta(\Delta\tau)\theta_\Delta(\tau, z, \varphi)|U_\Delta|V_\Delta = \Theta(\Delta\tau)\theta_\Delta(\tau, z, \varphi),$$

Following the argument of Lemma 4 in [19], we have that $\Theta(\tau)\theta(\tau, z, \varphi)$ is a non-holomorphic modular form of weight 2 on $\Gamma(4N)$. Note that $\Theta(\tau)$ is nowhere-vanishing on $\mathbb{H}$ and

$$\theta_\Delta(\tau + 1, z, \varphi) = \theta_\Delta(\tau, z, \varphi).$$

Thus, we complete the proof. □

We define a theta lift of f by

$$I_\Delta(\tau, f) = \int_M f(z)\theta_\Delta(\tau, z, \varphi). \tag{3.4}$$

Proposition 3.1 implies the convergence of the integral (3.4). Thus, we have $I_\Delta(\tau, f)$ is a (in general non-holomorphic) modular form of weight 3/2 on $\Gamma_1(N)$.

4. Proof of Theorem 1.1

Suppose that N is a positive integer and $\Gamma = \Gamma_0(N)$. By determining the Fourier expansion of $I_\Delta(\tau, f)$, we prove that the generating series for the twisted traces of CM values of a weakly holomorphic modular function on $\Gamma_0(N)$ is given by the holomorphic part of a harmonic weak Maass form of weight 3/2. Note that we have

$$\begin{aligned} I_\Delta(\tau, f) &= \int_M f(z) \sum_{X\in L} \chi_\Delta(X)\varphi^0(\sqrt{v}X, z)e^{2\pi i q(X)\tau} \\ &= \sum_{m\in\mathbb{Z}} \int_M \sum_{X\in L_{0,m}} \chi_\Delta(X) f(z)\varphi^0(\sqrt{v}X, z)e^{2\pi i q(X)\tau}, \end{aligned}$$

where

$$L_{0,m} = \{X \in L \,|\, q(X) = m\}.$$

If $m \neq 0$, then, since $\Gamma \setminus L_{0,m}$ is finite, we have

$$\begin{aligned} &\int_M \sum_{X\in L_{0,m}} \chi_\Delta(X) f(z)\varphi^0(\sqrt{v}X, z)e^{2\pi i q(X)\tau} \\ &\qquad = \sum_{X\in\Gamma\setminus L_{0,m}} \chi_\Delta(X) \int_M \sum_{\gamma\in\Gamma_X\setminus\Gamma} f(z)\varphi^0(\sqrt{v}X, \gamma z)e^{2\pi i q(X)\tau}. \end{aligned}$$

In the following proposition, we determine the mth Fourier coefficient of $I_\Delta(\tau, f)$ for $m \neq 0$.

Proposition 4.1. [3] *Let $X \in L_{0,m}$ and $\Gamma = \Gamma_0(N)$. Then we have the followings:*

(1) If $m>0$ and $X\in L_{0,m}$, then

$$\int_M \sum_{\Gamma_X\backslash\Gamma} f(z)\varphi^0(\sqrt{v}X,\gamma z) = \frac{1}{|\overline{\Gamma_0(N)_X}|} f(D_X).$$

(2) If $m<0$ and $m\notin -(\mathbb{Q}^\times)^2$ and $X\in L_{0,m}$, then

$$\int_M \sum_{\Gamma_X\backslash\Gamma} f(z)\varphi^0(\sqrt{v}X,\gamma z) \in L^1(M)$$

and

$$\int_M \sum_{\Gamma_X\backslash\Gamma} f(z)\varphi^0(\sqrt{v}X,\gamma z) = 0.$$

(3) If $m<0$ and $m\in -(\mathbb{Q}^\times)^2$, then

$$\int_M \sum_{\Gamma_X\backslash\Gamma} f(z)\varphi^0(\sqrt{v}X,\gamma z) \in L^1(M)$$

and

$$\int_M \sum_{\Gamma_X\backslash\Gamma} f(z)\varphi^0(\sqrt{v}X,\gamma z) = (a_{\ell_X}(0)+a_{\ell_{-X}}(0))\frac{1}{8\pi\sqrt{vm}}\beta(4\pi m) + \langle f, c(X)\rangle.$$

Recall that α_ℓ is the width of the cusp ℓ of Γ, and that $\sigma_\ell\in SL_2(\mathbb{Z})$ transforms the infinite cusp to ℓ. Then f has a Fourier expansion at the cusp ℓ of the form

$$f(\sigma_\ell z) = \sum_{n\in\frac{1}{\alpha_\ell}\mathbb{Z}} a_\ell(n)e(nz) \tag{4.1}$$

with $a_\ell(n)=0$ for $n\ll 0$.

Proposition 4.2. *If $m=0$, then we have*

$$\int_M \sum_{X\in L_{0,0}} f(z)\chi_\Delta(X)\varphi^0(\sqrt{v}X,z) = 0.$$

Proof. Let $Q_\ell = e(\sigma_\ell^{-1}z/\alpha_\ell)$ and $D_{1/T}=\left\{w\in\mathbb{C}\mid 0<|W|<\frac{1}{2\pi T}\right\}$ for $T>0$. We truncate M by setting

$$M_T = M - \coprod_{\ell\backslash Iso(V)} Q_\ell^{-1}D_{1/T}.$$

The regularized integral $\int_M^{reg}\sum_{X\in L_{0,0}} f(z)\chi_\Delta(X)\varphi^0(\sqrt{v}X,z)$ is defined by

$$\int_M^{reg} \sum_{X\in L_{0,0}} f(z)\chi_\Delta(X)\varphi^0(\sqrt{v}X,z) = \lim_{T\to 0}\int_{M_T}\sum_{X\in L_{0,0}} f(z)\chi_\Delta(X)\varphi^0(\sqrt{v}X,z).$$

Let X_ℓ be the primitive positive oriented vector in $L\cap\ell$. Note that $\chi_\Delta((\begin{smallmatrix}0&0\\0&0\end{smallmatrix}))=0$. Thus, we have by Stokes Theorem

$$\begin{aligned}\int_M \sum_{X\in L_{0,0}} f(z)\chi_\Delta(X)\varphi^0(\sqrt{v}X,z) &= \int_M^{reg} f(z)\sum_{X\in L_{0,0}}\chi_\Delta(X)\varphi^0(\sqrt{v}X,z)\\ &= \int_M^{reg} f(z)\sum_{\substack{\ell\in\Gamma\backslash Iso(V)\\ L_{0,0}\cap\ell\neq 0}}\sum_{X\in\ell\cap L_{0,0}}\chi_\Delta(X)\sum_{\gamma\in\Gamma_\ell\backslash\Gamma}\varphi^0(\sqrt{v}\gamma^{-1}X,z)\\ &= \sum_{\substack{\ell\in\Gamma\backslash Iso(V)\\ L_{0,0}\cap\ell\neq 0}}\sum_{\gamma\in\Gamma_\ell\backslash\Gamma}\int_M^{reg} f(z)\sum_{n=-\infty}^{\infty}\chi_\Delta(nX_\ell)\varphi^0(n\sqrt{v}X_\ell,\gamma z)\\ &= \frac{1}{2\pi i}\sum_{\substack{\ell\in\Gamma\backslash Iso(V)\\ L_{0,0}\cap\ell\neq 0}}\lim_{T\to 0}\int_{\partial M_T} f(z)\sum_{\gamma\in\Gamma_\ell\backslash\Gamma}\sum_{n=-\infty}^{\infty}\chi_\Delta(nX_\ell)\partial\xi^0(n\sqrt{v}X_\ell,\gamma z).\end{aligned}$$

Let $X_\ell^0=\left(\begin{smallmatrix}0&\beta_\ell\\0&0\end{smallmatrix}\right)$. We have by (3.1)

$$\partial\xi^0(\sqrt{v}X_\ell^0,gz)=\frac{-i}{(cz+d)^2 Im(gz)}e^{-\pi v r^2/Im(gz)^2}dz$$

for $g=(\begin{smallmatrix}a&b\\c&d\end{smallmatrix})\in SL_2(\mathbb{R})$. This implies that there is $\delta>0$ such that

$$|\partial\xi^0(\sqrt{v}X_\ell^0,gz)|\ll e^{-\delta y^2}dz$$

for all $g=(\begin{smallmatrix}a&b\\c&d\end{smallmatrix})\in SL_2(\mathbb{R})$ with $c\neq 0$, uniformly for $y>1$. Thus, we have

$$\begin{aligned}&\frac{1}{2\pi i}\sum_{\substack{\ell\in\Gamma\backslash Iso(V)\\ L_{0,0}\cap\ell\neq 0}}\lim_{T\to 0}\int_{\partial M_T} f(z)\sum_{\gamma\in\Gamma_\ell\backslash\Gamma}\sum_{n=-\infty}^{\infty}\chi_\Delta(nX_\ell)\partial\xi^0(n\sqrt{v}X_\ell,\gamma z)\\ &=\frac{-1}{2\pi i}\sum_{\substack{\ell\in\Gamma\backslash Iso(V)\\ L_{0,0}\cap\ell\neq 0}}\sum_{\ell'\in\Gamma\backslash Iso(V)}\lim_{T\to 0}\int_{z=iT}^{\alpha_{\ell'}+iT} f(\sigma_{\ell'}z)\sum_{\gamma\in\Gamma_\ell\backslash\Gamma}\sum_{n=-\infty}^{\infty}\chi_\Delta(nX_\ell)\partial\xi^0(n\sqrt{v}X_\ell^0,\sigma_\ell^{-1}\gamma\sigma_{\ell'}z)\\ &=\frac{-1}{2\pi i}\sum_{\substack{\ell\in\Gamma\backslash Iso(V)\\ L_{0,0}\cap\ell\neq 0}}\lim_{T\to 0}\int_{z=iT}^{\alpha_\ell+iT} f(\sigma_{\ell'}z)\sum_{n=-\infty}^{\infty}\chi_\Delta(nX_\ell)\partial\xi^0(n\sqrt{v}X_\ell^0,z)\\ &=\frac{1}{2\pi}\sum_{\substack{\ell\in\Gamma\backslash Iso(V)\\ L_{0,0}\cap\ell\neq 0}}\lim_{T\to 0}\int_{z=iT}^{\alpha_\ell+iT} f(\sigma_{\ell'}z)\sum_{n=-\infty}^{\infty}\chi_\Delta(nX_\ell)\frac{1}{y}e^{-\pi v(n\beta_\ell)^2/y^2}dx\\ &=\frac{1}{2\pi}\sum_{\substack{\ell\in\Gamma\backslash Iso(V)\\ L_{0,0}\cap\ell\neq 0}}\lim_{T\to 0}\frac{\alpha_\ell}{2\pi}a_\ell(0)\sum_{n=-\infty}^{\infty}\chi_\Delta(nX_\ell)\frac{1}{T}e^{-\pi v(n\beta_\ell)^2/T^2}dx.\end{aligned}$$

Note that for a fixed X_ℓ we have $\chi_\Delta(nX_\ell)$ is a Dirichlet character (see I.2 in [10]). Let R be the conductor of $\chi_\Delta(nX_\ell)$ and

$$g(\chi_\Delta(nX_\ell)) = \sum_{n \mod R} \chi_\Delta(n) e^{2\pi i n/R}.$$

Using twisted Poisson summation formula (see the formula (1.10) in [6]), we have

$$\int_M^{reg} f(z) \sum_{X \in L_{0,0}} \chi_\Delta(X) \varphi^0(\sqrt{v}X, z)$$
$$= \frac{\chi_\Delta(-X) g(\chi_\Delta(nX_\ell))}{R} \cdot \frac{a_\ell(0)(\alpha_\ell/\beta_\ell)}{2\pi\sqrt{v}} \lim_{T\to\infty} \sum_{w=-\infty}^{\infty} \chi_\Delta(wX) e^{-\pi w^2 T^2/(vR^2\beta_\ell^2)} = 0.$$

This completes the proof. □

Using Proposition 4.7 in [3], we immediately obtain by (3.3) that $\mathbf{t}_f(\chi_\Delta, -m^2)$ is zero for large $m > 0$.

Proposition 4.3. *Suppose that $f(z)$ is a weakly holomorphic modular function on $\Gamma_0(N)$ having the Fourier expansion as in (4.1). Then*

$$\mathbf{t}_f(\chi_\Delta, -m^2) = 0 \quad \text{for } m \gg 0.$$

Proposition 4.1, 4.2 and 4.3 immediately give the Fourier expansion of $I_\Delta(\tau, f)$.

Theorem 4.4. *Let $f \in M_0^!(\Gamma_0(N))$ with Fourier expansion as in (4.1) and $\tau = u + iv \in \mathbb{H}$. If the constant coefficient of f at each cusp of M vanishes, then $I_\Delta(\tau, f)$ is a weakly holomorphic modular form of weight 3/2 for $\Gamma_1(N)$. The Fourier expansion of $I_\Delta(\tau, f)$ is given by*

$$I_\Delta(\tau, f) = \sum_{\substack{m\in\mathbb{Z} \\ m \gg -\infty}} \mathbf{t}_f(\chi_\Delta; m) q^m.$$

If the constant coefficient of f does not vanish, then $I_\Delta(\tau, f)$ is non-holomorphic, and in the Fourier expansion the following terms occur in addition:

$$\sum_{m>0} \sum_{X \in \Gamma_0(N)\backslash L_{0,-m^2}} \frac{a_{\ell_X}(0) + a_{\ell_{-X}}(0)}{8\pi\sqrt{v}m} \chi_\Delta(X) \beta(4\pi v m^2) q^{-m^2}.$$

Now we prove Theorem 1.1.

Proof of Theorem 1.1. Suppose that $I_\Delta(\tau, f)$ has the Fourier expansion of the form

$$I_\Delta(\tau, f) = \sum_m a_m(v) e^{2\pi i m u}.$$

Let

$$F_t(\tau) = \frac{1}{2} \sum_m \left(\frac{m}{t}\right)^2 a_m(v) e^{2\pi i m u} - \frac{1}{2}\left(\frac{-1}{t}\right) \sum_m \left(\frac{m}{t}\right) a_m(v) e^{2\pi i m u}.$$

Theorem 4.4 implies that $F_t(\tau)$ is a weakly holomorphic modular form of weight $3/2$ on $\Gamma_1(4t^2N)$, which has the Fourier expansion of the form

$$F_t(\tau) = \sum_{\substack{m>m_0 \\ \left(\frac{m}{t}\right)=-1}} \mathbf{t}_f(\chi_\Delta; m) q^m. \tag{4.2}$$

Note that we can take m_0 independent of t. Thus, if r is sufficiently large, then

$$F_t(\tau)|U_r = 2 \sum_{\substack{m>0 \\ \left(\frac{m}{t}\right)=-1}} \mathbf{t}_f(\chi_\Delta; rm) q^m.$$

This completes the proof. $\square$

5. Proof of Theorem 1.2

We begin by stating the following lemma.

Lemma 5.1. [20, Theorem 1.1] *Suppose that p is an odd prime, and that k and m are integers with k odd. Let N be a positive integer with $4|N$ and $p \nmid N$. Let $g(\tau) = \sum a(n)q^n \in M^!_{\frac{k}{2}}(\Gamma_0(N)) \cap \mathcal{O}_K((q))$, where $\mathcal{O}_K$ denotes the ring of integers of an algebraic number field K. If m is sufficiently large, then for each positive integer ν, a positive proportion of primes $r \equiv -1 \pmod{Np^\nu}$ have the property that*

$$a(r^3p^mn) \equiv 0 \pmod{p^\nu}$$

for all n relatively prime to rp.

To use Lemma 5.1 in our case, we have to show that traces of singular moduli are algebraic numbers.

Lemma 5.2. *If $f(\tau) \in M^!_0(\Gamma_0(N)) \cap K((q))$, then $\boldsymbol{t}_f(\chi_\Delta, m)$ is an algebraic number for every integer m.*

Proof. Assume that m is a positive integer. Following the argument of [7, Lemma 5.2], we have that $f(\tau_0)$ is an algebraic number for a CM point τ_0. Thus, $\mathbf{t}_f(\chi_\Delta, m)$ is also an algebraic number by (2.1). In case when m is not positive, we can easily see that $\mathbf{t}_f(\chi_\Delta, m)$ is an algebraic number as well from (2.2) and (2.3). $\square$

Now we prove Theorem 1.2.

Proof of Theorem 1.2. Let

$$F_t(\tau) = 2 \sum_{\substack{m>m_t \\ \left(\frac{m}{t}\right)=-1}} \mathbf{t}_f(\chi_\Delta; m) q^m.$$

From (4.2) we have $F_t(\tau)$ is a weakly holomorphic modular form of weight 3/2 on $\Gamma_1(4t^4N)$. Since

$$\Delta(\tau) := q\prod_{n=1}^{\infty}(1-q^n)^{24}$$

is the cusp form of weight 12 on $SL_2(\mathbb{Z})$, for sufficiently large m, $\Theta(\tau)\Delta^m(\tau)F_t(\tau)$ is a cusp form, and its Fourier coefficients are algebraic numbers. Hence, any $\mathbb{Z}$-module generated by the Fourier coefficients of $\Theta(\tau)\Delta^m(\tau)F_t(\tau)$ is finitely generated [18, Theorem 3.52]. Thus, there exists an integer μ such that

$$\mu F_t(\tau) \in M^!_{\frac{3}{2}}(\Gamma_1(4t^2N)) \cap \mathcal{O}_K((q)), \tag{5.1}$$

where K is the field generated by Fourier coefficients of $F_t(\tau)$. Note that Lemma 5.1 can be extended to $\Gamma_1(N)$ (See [5]). Thus, from Lemma 5.1 and (5.1) we complete the proof. $\square$

Acknowledgement

This work was supported by the Korea Research Foundation Grant funded by the Korean Government (KRF-2008-331-C00005). The author wishes to express his gratitude to KIAS for its support through Associate membership program.

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